Materials Science Research india Vol. **4**(1), 19-24 (2007)

On the model reduction of discrete time-varying systems

A. S. AL-FHAID

Department of Mathematics, College of Sciences, P.O. Box 80203,
King Abdulaziz University, Jeddah - 21589 (Saudi Arabia)

(Received: May 10, 2007; Accepted: June 10, 2007)

ABSTRACT

A method for approximating multivariable linear discrete time-varying system is described. The method is based on minimizing the difference between the system outputs and the reduced order model outputs. The optimal model parameters are given by a set of non-linear equations. It is shown that when the output and the state of the model are equal, the synthesis simplifies to a linear problem. The results are illustrated by a numerical example.

Key words: Model reduction, discrete time-varying system.

INTRODUCTION

In many areas of engineering the approximation of high order linear systems requires order reduction due to real time computation, speed and memory constraints.

Extensive research efforts devoted to the theory of model reduction of linear time-invariant systems can be found[1]. However, less attention has been directed to the problem of model reduction of time-varying linear systems[2-7]. This is surprising, because reduced model systems operating under varying conditions, is needed in many applications, such as large space structures, flexible flight systems, nuclear reactors, microphone transmitter which contain a variable resistor and parametric amplifiers which derive their gain from variable-reactance circuit elements. In addition, problems involving control of missile flights require inclusion of time varying parameters whereby the responses are characterized by linear differential equations with time-varying aerodynamic coefficients. In adaptive control systems, the parameters of the controller are adjusted to compensate for changes in process dynamics stemming from changes in the environment.

In the present investigation, a method for approximating multivariable linear discrete time-varying system is described. The method considered is similar to the method presented by[4] who minimize the error between the outputs of the original system and the reduced order model. The optimal reduced model parameters are given by a set of nonlinear equations. It is also shown that, when the output and the state of the reduced model are equal,, the synthesis simplifies to a linear problem. A numerical example is provided to illustrate the approach.

Problem Statement

Consider the following linear. discrete time-varying system:

$$X(k+1) = A(k)X(k) + B(k)W(k) \quad \text{...(1)}$$

$$Y(k) = C(k)X(k) \quad \text{...(2)}$$

where

and $Y \in R^m, A(\cdot), B(\cdot),$

and $C(\cdot)$ are matrices of appropriate size is zero mean Gaussian source with covariance $\phi(k)$. The initial state is assumed to be Gaussian with mean X_0 and covariance V_0.

It is desired to find an optimal model with $\mathrm{m} \le \mathrm{r} < \mathrm{n}$ of the form:

...(3)

$$Y_r(k) = C_r(k) X_r(k). \qquad ...(4)$$

Where

$X_r \in R^r$, $Y_r \in R^m$ and such that the performance measure defined by:

$$J_k = \frac{1}{2} E[e(k)^T Q(k)^T e(k)] \qquad ...(5)$$

is minimum, $Q(k) = Q(k)^T > 0$ and it is assumed that $X(0), X_r(0), W(k)$ are mutually independent.

RESULTS AND DISCUSSION

The augmented system state and the reduced model state may be expressed as

$$Z(k+1) = F(k)Z(k) + G(k)W(k) \qquad ...(6)$$

where

...(7)

Define the second-moment matrix

$$P(k) = E[Z(k)Z^T(k)] \qquad ...(8)$$

$P(k)$ is the solution of matrix difference equation.

$$P(k+1) = F(k)P(k)F^T(k) + G(k)\phi(k)G^T(k)$$
...(9)

$$P(0) = P_0$$

The reduction error is defined as

$$e(k) = Y(k) - Y_r(k) = L(k)Z(k).$$
...(10)

Where

$$L(k) = [C(k) - C_r(k)]. \qquad ...(11)$$

Hence, the mean square tracking error given by (5) becomes

$$J_k = \frac{1}{2} tr[V(k)P(k)]. \qquad ...(12)$$

Where

$$V(k) = L^T(k)Q(k)L(k)$$
...(13)

By optimizing (12) with respect to the free parameters $A_r(k), B_r(k),$ and $C_r(k)$ we have the following theorem:

$$X_r(k+1) = A_r(k)X_r(k) + B_r(k)W(k)$$

$$F(k) = \begin{bmatrix} A(k) & 0 \\ 0 & A_r(k) \end{bmatrix}, \quad G(k) = \begin{bmatrix} B(k) \\ B_r(k) \end{bmatrix}.$$

Theorem 1

In order that $A_r(k).B_r(k),$ and $C_r(k)$ be optimal, it is necessary that

$$A_r(k) = M^{-1}(k+1)N(k+1)A(k)P_{12}(k)P_{22}^{-1}(k)$$
...(14)

$$B_r(k) = M^{-1}(k+1)N(k+1)B(k)$$
...(15)

$$C_r(k) = C(k)P_{12}(k)P_{22}^{-1}(k). \qquad ...(16)$$

Where

$$M(k+1) = C_r^T(k+1)Q(k+1)C_r(k+1)$$
...(17)

$$N(k+1) = C_r^T(k+1)Q(k+1)C(k+1)$$
...(18)

$$P_{12}(k+1) = A(k)P_{12}(k)A_r^T(k) + B(k)\phi(k)B_r^T(k)$$
...(19)

$$P_{22}(k+1) = A_r(k)P_{22}(k)A_r^T(k) + B_r(k)\phi(k)B_r^T(k)$$
...(20)

(Proof: see the appendix)

Remarks

1. The matrix $P_{22}(k+1)$ is positive definite. This condition guarantees the invertibility of $P_{22}(k+1)$.
2. In order $A_r(k)$ and $B_r(k)$ exist, it should be that $M(k+1)$ positive definite.
3. If $M(k+1)$ is not positive definite, one can find the pseudo-inverse of $M(k+1)$.
4. The general set of necessary conditions (14)-(20) are nonlinear and therefore requires the use of iterative numerical algorithm for its solution.
5. One can vary the weighting matrix $Q(k)$ according to the relative importance of the tracking error.

Special Case

Consider the problem of finding reduced model for system with $C_r(k) = I_r$, i.e., $Y_r(k) = X_r(k)$. The only remaining is now the matrices $A_r(k)$ and $B_r(k)$. The following results are obtained.

Theorem 2

Suppose $C_r(k) = I_r$. Then there exist an optimal reduced order model such that

$$A_r(k) = C(k+1)A(k)P_{12}(k)P_{22}^{-1}(k) \quad ...(21)$$

$$B_r(k) = C(k+1)B(k) \quad ...(22)$$

where $P_{12}(k+1)$ and $P_{22}(k+1)$ are given by (19) and (20)

The proof of Theorem 2 is directly obtained from the proof of theorem 1 by setting $C_r(k) = I_r$. Thus the problem of optimum model reduction now simplified drastically to the solution of linear matrix equations. The main computations can be summarized as follows:

Step 1: Input the maximum value of k.

Step 2: Set $k = 0$, Choose

$$X(0), X(0), P_{12}(0) \text{ and } P_{22}(0).$$

Step 3: Solve for $A_r(k)$ and $B_r(k)$ using (21)-(22).

Step 4: Solve for $P_{12}(k+1)$ and $P_{22}(k+1)$ using (19)-(20).

Step 5: Set $k = k+1$ and return to step 3.

Step 6: Terminate if $k = k_{\max}$.

Example

To illustrate this technique, consider the following system

$$X(k+1) = A(k)X(k) + B(k)W(k) \quad ...(23)$$

$$...(24)$$

Where

$$Y(k) = C(k)X(k)$$

$$A(k) = \begin{bmatrix} 0.9 & 0 & 1/(k+1) \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

$$B^T(k) = \begin{bmatrix} 1/(k+1) & 1 & 0 \end{bmatrix},$$

$$C(k) = \begin{bmatrix} 1 & 0.05 & 0 \end{bmatrix}$$

It is desired to generate a first order model of the form.

$$Y_r(k+1) = A_r(k)Y_r(k) + B_r(k)W(k)$$

using the results of Theorem 2 with $X(0) = [0 \quad 0 \quad 0]^T$, $X_r(0) = 0$.

$P_{12}(0) = [0 \quad 0 \quad 0]^T$, $P_{22}(0) = 1$, and $Q(k) = \phi(k) = 1$, we get the following values of $A_r(k)$ and $B_r(k)$ as shown in Fig.1 and Fig.2.

The impulse response of the original third order system and the optimal first order reduced model are shown in Fig.3. It is seen that the method yields a good approximation to the original impulse response.

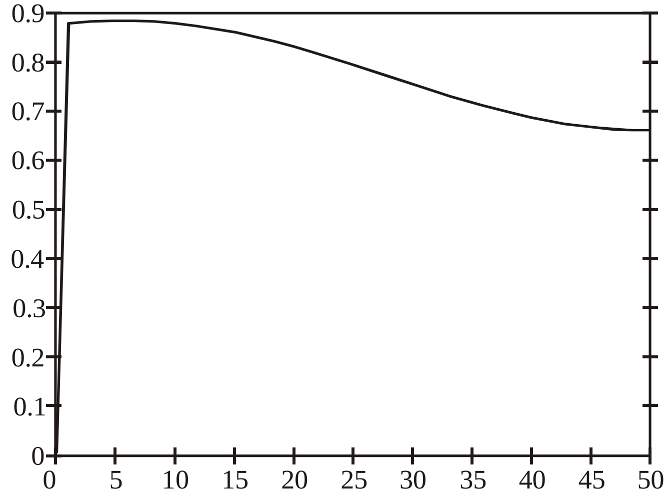

Fig. -1: Ar(K)

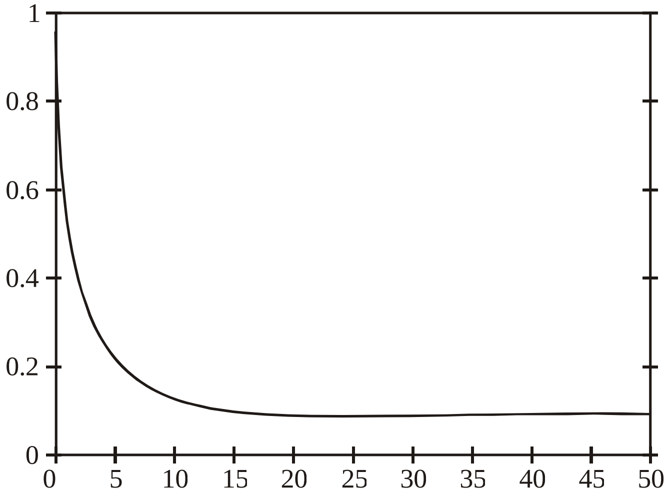

Fig. -2: Br(K)

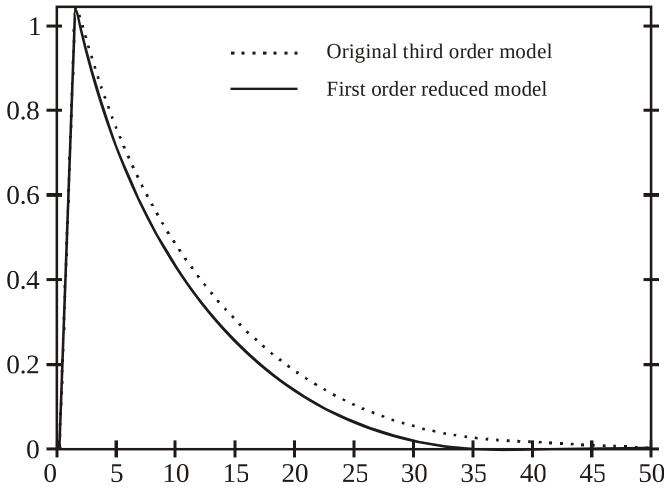

Fig. -3: Impulse response plots

CONCLUSION

A method is proposed for designing an optimal reduced model for discrete time varying systems. It has been shown this method requires the solution of nonlinear matrix equations. However, when the output and the state of the reduced model are equal, the problem become linear, and an explicit solution is possible by solving simple linear matrix equations. The model can be designed to be more accurate at some time than at other times by proper selection of the weighting matrices. The usefulness of the method was illustrated by an example. It should be pointed out that the method can also be applied to the time-invariant case. Presently, we are investigating a simple method for the reduction of continuous time varying system.

REFERENCES

1. Bamani, G. Al-Saddki, and M. S. Fadali. Robust model reduction of discrete stochastic systems. ISMM *International Conference on Computer Application.* **1,** 232-134 (1989).
2. Rao, A.S., S. Lamba, and S.Rao. Application of routh approximation method for reducing the order of a class of time varying systems. IEEE Transaction on Automatic Control, **AC-25,** 110-113 (1980).
3. Baram, Y., and G. Kalit. Order reduction in linear state estimation under performance constraints, IEEE Transaction on Automatic Control, **AC-32,** 983-989 (1987).
4. Bailey, M., and C. Sims. A procedure for designing time variable reduced order models. *Proceeding of 24th Conference on Decision and Control,* **1,** 593-594 (1985).
5. Benmahammed, K. Model reduction of uniformly controllable continuous time varying linear systems. *American Control Conference,* **3,** 1500-1503 (1987).
6. Zhu, J., and C. D. Johnson. New results in the reduction of linear time varying dynamical systems. *Proceeding of 26th Conference on Decision and Control.* **1,** 129-135 (1987).
7. Bamani, A., M. Al-fhaid, and G. Lee. Optimal direct minimal order dynamic compensators for time-varying linear systems. Accepted for presentation in the IASTED International Conference on Adaptive Control and Signal Processing, Oct. 10-12, 1990, U.S.A (1990).
8. Kwakernaak, H., and R. Sivan. Linear Optimal Control Systems. John Wiley and Sons, Inc., New York. 528-521 (1972).
9. Erwin Kreyszig. Advanced Engineering Mathematics.8 th Edition . John Wiley and Sons, Inc., New York..
10. Steven C. Chapra, and Raymond P.Canale. Numerical Methods for Engineers. 2 nd Edition. McGraw- Hill International Editions, Applied Mathematics Series (1999).

APPENDIX

Proof of Theorem 1

The objective is to select matrix sequences $\{A(k)\},\{B(k)\}$ and $\{C(k)\}$ to minimize the performance measure (12) subject to (9). This is a difficult problem when solved directly using the minimum principle as noted in [4]. However, the problem is simplified if one only considers optimization of the free parameters at the next stage [8]. Let us now consider the problem of minimizing.

$$J_{k+1} = \frac{1}{2} tr\left[V(k+1)P(k+1)\right] \qquad ...(25)$$

The necessary conditions for optimality are:

$$\frac{\delta J_{k+1}}{\delta A_r(k)} = M(k+1)A_r(k)P_{22}(k) - N(k+1)A(k)P_{12}(k) = 0$$

...(26)

$$\frac{\delta J_{k+1}}{\delta B_r(k)} = M(k+1)B_r(k)\phi(k) - N(k+1)B(k)\phi(k) = 0$$

...(27)

$$\frac{\delta J_{k+1}}{\delta C_r(k+1)} = C_r(k+1)P_{22}(k+1) - C(k+1)P_{22}(k+1) = 0$$

...(28)

where is partitioned as

$$\begin{bmatrix} P_{11}(k) & P_{12}(k) \\ P_{12}^T(k) & P_{22} \end{bmatrix}$$

...(29)

and

$$M(k+1) = C_r^T(k+1)Q(k+1)C_r(k+1)$$

...(30)

$$N(k+1) = C_r^T(k+1)Q(k+1)C(k+1)$$

...(31)

Substituting (7) and (29) in (9), we get (19) and (20). Equations (14)-(16) can be obtained directly from (26)-(28). This completes the proof.