

Positive solution for fourth order boundary value problem

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ABSTRACT

In this paper, we investigate the problem of existence of positive solutions for the nonlinear fourth order boundary value problem:

$$\begin{aligned} D^4 u(t) &= \lambda a(t)f(u(t)), \quad 0 < t < 1, \\ u(0) &= u''(0) = u'(1) = u'''(1) = 0, \end{aligned}$$

where λ is a positive parameter. By using Krasnoesel'skii's fixed point theorem of cone, we establish various results on the existence of positive solutions of the boundary value problem. Under various assumptions on $a(t)$ and $f(u(t))$, we give the intervals of the parameter λ which yield the existence of the positive solutions.

Key words: Fourth order boundary value problem, Krasnoesel'skii's fixed point theorem, Green's function, Positive solution.

INTRODUCTION

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoesel'skii's theorem on cone expansion and compression and its norm-type version due to Guo and Lakshmikantham¹¹. Very recently, Li¹² used Krasnoesel'skii's fixed point theorem to prove some existence results to the nonlinear third singular boundary value problem:

$$\begin{aligned} u'''(t) + \lambda a(t)f(u(t)) &= 0, \quad 0 < t < 1, \\ u(0) &= u'(0) = u''(1) = 0. \end{aligned}$$

In fact, the problem of existence of positive solutions for the nonlinear fourth order boundary value problem has been studied by many authors under various boundary conditions and by different approaches. For some other results, We refer the reader to papers of Agarwal¹, Agarwal, O'Regan and Wong², Avery, Chyan and Henderson³, Chyan

and Henderson⁷, Dalmasso⁸ and Guo and Lakshmikantham¹¹. The purpose of this paper is to establish the existence of positive solutions to nonlinear fourth order boundary value problem:

$$D^4 u(t) = \lambda a(t)f(u(t)), \quad 0 < t < 1, \quad \dots(1)$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0, \quad \dots(2)$$

where λ is a positive parameter. Throughout the paper, we assume that

C1: $f : [0, \infty) \rightarrow [0, \infty)$ is continuous

C2: $a : (0, 1) \rightarrow [0, \infty)$ is continuous function

such that $\int_0^1 a(t)dt > 0$.

Here, by a positive solution of the boundary value problem we mean a function which is positive on $(0, 1)$ and satisfies differential equation (1) and

the boundary condition (2).

Definition 1.

Let E be a real Banach space. A nonempty closed set $K \subset E$ is called **cone** of E if it satisfies the following conditions:

1. $x \in K, \lambda \geq 0$ implies that $\lambda x \in K$
2. $x \in K, -x \in K$ implies that $x = 0$.

Definition 2

An operator is called **completely continuous** if it is continuous and maps bounded sets into precompact sets.

The Green's function $G : [0,1] \times [0,1] \rightarrow [0,\infty)$ for the problem (1) – (2) is

$$G(t,s) = \begin{cases} \frac{(1-(1-s)^2)}{2} t - \frac{t^2}{6}, & \text{if } 0 \leq t \leq s \leq 1; \\ \frac{((1-s)^2-1)}{2} t - \frac{t^2}{6} + \frac{(t-s)^2}{6}, & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Then problem (1) – (2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) a(s) f(u(s)) ds, \quad 0 \leq t \leq 1. \quad \dots(3)$$

It is easy to verify that G is a continuous function, and $t, s \in (0,1)$. We will need the following fixed point theorem, which is due to Krasnosel'skii¹⁰, to prove some of our results.

Theorem 1.

Let E be a Banach space and $K \subset E$ is a cone in E . Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$. Let

$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be completely continuous operator. In addition suppose either:

H1:

$\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$ or

H2:

$\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$.

holds. Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Solution in the cone

In this section, we will apply Krasnosel'skii's fixed point theorem to the eigenvalue problem (1) – (2). We note that $u(t)$ is a solution of (1) – (2) if and only if

$$u(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s)) ds, \quad 0 \leq t \leq 1.$$

Now, we consider the Banach space $E = C[0,1]$ equipped with standard norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|, \quad u \in E.$$

We define a cone P by

$$P = \left\{ u \in E : \min_{0 \leq t \leq 1} u(t) \geq \frac{m}{M} \|u\| \right\},$$

where $m = \min_{0 \leq s \leq 1} \{G(t,s), 0 \leq t \leq 1\}$ and $M = \max_{0 \leq s \leq 1} \{G(t,s), 0 \leq t \leq 1\}$ and define an

integral operator $T : P \rightarrow E$ by

$$Tu(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s)) ds.$$

It is clear that if (C1) and (C2) hold, then is a completely continuous operator. Following Sun and Wen¹³, we define some important constants:

$$F_0 = \limsup_{u \rightarrow 0^+} \frac{f(u)}{u}, \quad f_0 = \liminf_{u \rightarrow 0^+} \frac{f(u)}{u}, \\ F_\infty = \limsup_{u \rightarrow \infty} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u \rightarrow \infty} \frac{f(u)}{u}$$

Here we assume that $\frac{1}{Mf_\infty} = 0$, if $f_\infty \neq \infty$,

$\frac{1}{mF_0} = \infty$, if $F_0 = 0$, $\frac{1}{Mf_0} = 0$, if $f_0 \neq \infty$ and

We assume that $A = \int_0^1 a(s) ds$.

Lemma 1. $T(P) \cap P \neq \emptyset$.

Proof. Since $G(t, s) \geq 0$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t, s) a(s) f(u(s)) ds \\ &\leq \lambda m \int_0^1 a(s) f(u(s)) ds \\ &\leq \lambda m \int_0^1 \frac{G(t, s)}{M} a(s) f(u(s)) ds \\ &\leq \frac{m}{M} Tu(t), \quad \text{for all } t \in [0, 1]. \end{aligned}$$

So, $\min_{0 \leq t \leq 1} Tu(t) \geq \frac{m}{M} \|Tu\|$ and then $Tu \in P$. Thus

The proof is complete. $\square$

Theorem 2.

Suppose that C1 and C2 holds. If

$MAf_{\gamma} > mAf_0$, then for each $\lambda \in \left(\frac{1}{MAf_{\gamma}}, \frac{1}{mAf_0} \right)$ the problem (1) and (2) has at least one positive solution.

Proof.

By definition of f_0 we can see that there exists an $I_1 > 0$, such that $f(u) \leq (F_0 + \varepsilon)u$

for $0 < u < I_1$. If $\varepsilon > 0$ sufficiently small such that

$$\begin{aligned} \|Tu(t)\| &= \max_{0 \leq t \leq 1} \lambda \left| \int_0^1 G(t, s) a(s) f(u(s)) ds \right| \\ &\leq \lambda M \int_0^1 a(s) f(u(s)) ds \\ &\leq \lambda M (F_0 + \varepsilon) \|u\| \int_0^1 a(s) ds \\ &\leq \lambda M (F_0 + \varepsilon) \|u\| A, \end{aligned}$$

where

We choose $\varepsilon > 0$ sufficiently small such that

Then we have $\|Tu\| \leq \|u\|$. Thus if we let $\Omega_1 = \{u \in E : \|u\| < I_1\}$, for $u \in P \cap \Omega_1$.

Following Sun and Wen¹³, we choose CE (α, Y) such that

$$\lambda M \int_0^1 (f_{\gamma} - \varepsilon) \int_0^1 a(s) f(u(s)) ds \geq 1,$$

where $\gamma \in (3/4, 1)$. There exists $I_2 > 0$, such that

for $u > I_2$. If

we have

$$\begin{aligned} \|Tu(t)\| &= \max_{0 \leq t \leq 1} \lambda \left| \int_0^1 G(t, s) a(s) f(u(s)) ds \right| \\ &\leq \lambda m \int_0^1 a(s) f(u(s)) ds \\ &\leq \lambda m (f_{\gamma} - \varepsilon) \|u\| \int_0^1 a(s) ds \\ &\leq \lambda m (f_{\gamma} - \varepsilon) \|u\| A. \end{aligned}$$

We choose ε sufficiently small such that

Then we have

Thus if we let $\Omega_2 = \{u \in E : \|u\| < I_2\}$, then $\|Tu\| \leq \|u\|$ for $u \in P \cap \Omega_2$.

Condition (H1) of Krasnosel'skii's fixed point theorem is satisfied. So there exists a fixed point of T in P . This complete the proof.

Theorem 3.

Suppose that C1 and C2 holds. If $mAf_0 > MAf_{\gamma}$, then for each

$\lambda \in \left(\frac{1}{MAf_{\gamma}}, \frac{1}{mAf_0} \right)$ the problem (1) and (2) has at least one positive solution.

Proof.

By definition of f_0 we can see that there exists an $I_1 > 0$, such that $f(u) \leq (f_0 + \varepsilon)u$

for $0 < u < I_1$. If $\varepsilon > 0$ sufficiently small such that

$$\begin{aligned} \|Tu(t)\| &= \max_{0 \leq t \leq 1} \lambda \left| \int_0^1 G(t, s) a(s) f(u(s)) ds \right| \\ &\leq \lambda m \int_0^1 a(s) f(u(s)) ds \\ &\leq \lambda m (f_0 + \varepsilon) \|u\| \int_0^1 a(s) ds \\ &\leq \lambda m (f_0 + \varepsilon) \|u\| A. \end{aligned}$$

where $A = \int_0^1 a(s) ds$.

We choose ε sufficiently small such that $(f_0 - \varepsilon)\lambda Am^3 > 1$. Then we have $\|Tu\|^3 \leq \|u\|$, for $u \in P \cap C^1(\Omega_1)$. By the same method in the proof of theorem 2, we can see that, if $u \in P$ with $\|u\| = l_2$, $\|Tu\| \leq \|u\|$, for $u \in P \cap C^1(\Omega_2)$. Condition (H2) of Krasnosel'skii's fixed point theorem is satisfied. So there exists a fixed point of T in P. This complete the proof.

Theorem 4. Suppose that $\lambda M A f(u) < u$ for $u \in (0, \frac{1}{M})$. Then the problem (1) and (2) has no positive solution.

Proof. Following Sun and Wen¹³, we assume to the contrary that u is a positive solution of (1) and (2). Then

$$u(1) = \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds < \frac{1}{AM} \int_0^1 G(1,s) a(s) u(s) ds \\ \leq \frac{u(1)}{A} \int_0^1 a(s) ds = u(1).$$

This is a contradiction and complete the proof.

Theorem 5. Suppose that $\lambda M A f(u) > u$ for $u \in (0, \frac{1}{M})$. Then the problem (1) and (2) has no positive solution.

Proof. Assume to the contrary that u is a positive solution of (1) and (2). Then

$$u(1) = \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds > \frac{1}{m\lambda} \int_0^1 G(1,s) a(s) u(s) ds \\ \geq \frac{u(1)}{A} \int_0^1 a(s) ds = u(1).$$

This is a contradiction and complete the proof.

Example 1. Consider the equation

$$D^4 u(t) = \lambda \frac{9}{2} t^2 \frac{(u(t) + 9u^2(t))}{(1+u(t))} (8 + \cos u), \quad 0 \leq t \leq 1, \\ u(0) = u'(0) = u'(1) = u''(1) = 0, \quad (5)$$

where $\lambda > 0$ is a parameter, together with the boundary conditions (2). We consider $a(t) = \frac{9}{2} t^2$ and

$$f(u) = \frac{(u + 9u^2)}{(1+u)} (8 + \cos u). \quad \text{Then}$$

$$F_0 = f_0 = 9, F_{\frac{1}{2}} = 72, f_{\frac{1}{2}} = 63 \quad \text{and}$$

$$9u < f(u) < 72u. \quad \text{We assume that } m = 0.123$$

$$\text{and } M = 0.333. \quad \text{From theorem 2, we see that if}$$

$$\lambda \in (0.0143, 0.037), \quad \text{then the problem (2), (4)}$$

$$\text{has a positive solution. By theorem 4, we have that}$$

$$\text{if } \lambda < 6.5, \quad \text{then the problem (2), (4) has no positive}$$

$$\text{solution. By theorem 5, we have that if } \lambda > 2.34,$$

$$\text{then the problem (2), (4) has no positive solution.}$$

$\varepsilon > 0$

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