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Different choices of coset representatives

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ABSTRACT

In this paper, we show that for the algebras $\underline{A}$ and A, the map $\theta: \underline{A} \to A$ defined by $\theta_{\underline{A}}(\delta_s \otimes u) = \delta_{s \quad \gamma}(\underline{a}) \otimes \gamma(\underline{a})^{-1}u$, where $(\delta_s \otimes u) \in \underline{A}$ and $\underline{a} = \{\delta_s \otimes u\}$, is a morphism in the category C. Moreover, it is shown that the morphism $\theta : \underline{A} \to A$ is an algebra map.

Key words: Finite group factorization, Algebras, Bicrossproduct Algebras.

INTRODUCTION

The fact that a Hopf algebra H = $\Bbbk M \bowtie \Bbbk(G)$ can be constructed for every factorization X = GM of a group into two subgroups G and M, is well known. This bicrossproductconstruction is one of the sources of true non-commutative and non-cocommutative Hopf algebras[7-10].

These bicrossproduct Hopf algebras have been studied intensively[1-6]. In[3], Beggs has considered more general factorizations of groups, and their corresponding algebras. More specifically, it was shown that it is possible to construct a non-trivially associated tensor category $\mathcal{C}$ from data which is a choice of left coset representatives *M* for a subgroup G of a finite group X. The objects of this category are the right representations of G that possess M-grading. The group action and the grading in the definition of $\mathcal{C}$ were combined by considering a single object A spanned by a basis $\delta_s \otimes u$ for S $\in$ M and $u \in$ G. This object, A, was shown to be an algebra in $\mathcal{C}$ under certain conditions.

In this paper, we show that for the algebras $\underline{A}$ and A, the map $\theta: \underline{A} \to A$ defined by $\theta_{\underline{A}}(\delta_s \otimes u) = \delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1}u$, where $(\delta_s \otimes u) \in \underline{A}$ and a = $\langle \delta_s \otimes u \rangle$, is a morphism in the category $\mathcal{C}$. Moreover, it is shown that the morphism $\theta: \underline{A} \to A$ is an algebra map.

$$s \triangleright (t \triangleright u) = \tau(s,t)((s.t) \triangleright u)\tau(s \triangleleft (t \triangleright u), t \triangleright u)^{-1},$$

We use the same formulas and ideas given in[3], which is based on the papers[4,5]. Throughout the paper we assume that all mentioned groups are finite, and that all vector spaces are finite dimensional over a field $\Bbbk$.

Preliminaries

In order to make the paper self contained, we include the following definitions and results of[3].

Definition

For a group X, consider the factorization X = GM where G is a subgroup of X and M$\subset$X is a set of left coset representatives. For $\chi \in$ X the factorization $\chi = us$ for $u \in$ G and $s \in$ M is unique. Define τ(s, t) $\in$ G and s $\cdot$ t $\in$ M as well as the functions : M $\times$ G $\to$ G and : M $\times$ G $\to$ M by the unique factorizations in X: st = τ(s, t)(s $\cdot$ t) and su = (s u) (s u) for s, s $u \in$ M and u, s $u \in$ G.

For t, s, p $\in$ M and u, $\upsilon \in$ G, the following identities hold:

$$(s.t) \triangleleft u = (s \triangleleft (t \triangleright u)).(t \triangleleft u), s \triangleright uv = (s \triangleright u)((s \triangleleft u) \triangleright v) \;,$$

$$(p \triangleleft \tau(s,t)).(s.t) =) p.s).t,$$
$$e \triangleleft v = e, e \triangleright v = v, t \triangleright e = e, t \triangleleft e = t. \quad ...(1)$$

The category $\mathcal{C}$ is defined to be a category of finite dimensional vector spaces over a field k, whose objects are right representations of the group G and have G\X-gradings, i.e., an object V can be written as $\oplus_{s\in G\backslash X} V_s$. The action for the representation is written as :V ×G →V . In addition, it is supposed that the action and the grading satisfy the compatibility condition, i.e., $\langle \xi \quad u\rangle = \langle \xi \rangle \quad u$. The morphisms in the category $\mathcal{C}$ is defined to be linear maps which preserve both the grading and the action, i.e., for a morphism ϑ : V → W we have $\langle \vartheta(\xi)\rangle = \langle \xi \rangle$ and $\vartheta(\xi) \quad u = \vartheta(\xi \quad u)$ for all $\xi \in$ V and $u \in$ G. $\mathcal{C}$ is a tensor category with actions and gradings given by

$$\langle \xi \otimes \eta \rangle = \langle \xi \rangle . \langle \eta \rangle \text{ and } \langle \xi \otimes \eta \rangle \bar{\triangleleft} u = \xi \bar{\triangleleft} \left(\langle \eta \rangle \triangleright u \right) \otimes \eta \bar{\triangleleft} u$$

There is an associator Φ_{UVW} : (U⊗V)⊗ W → U⊗(V⊗ W) given by

$$\Phi\left((\xi \otimes \eta) \otimes \zeta \right) = \xi \bar{\triangleleft} \tau\left(\langle \eta \rangle , \langle \zeta \rangle \right) \otimes (\eta \otimes \zeta).$$

The dual object of V$\in \mathcal{C}$ is of the form $V^* = \oplus_{s\in M} V^*_{s^L}$, and we define $\langle a \rangle = s^L$ when $a \in V^*_{s^L}$. The evaluation map ev : V*⊗ V→ k is defined by ev(a,ξ) = a(ξ) = (a (⟨ξ⟩ u))(ξ u), or if we put η=ξ u we get a(η u^{-1})=(a ((⟨η⟩ u^{-1}) u))(η)= (a (⟨η⟩ u^{-1})$^{-1}$(η). If this is rearranged to give a u, we get the following formula:

$$(a \bar{\triangleleft} v)(\eta) = \alpha\left(\eta \bar{\triangleleft} \tau\left(\langle \eta \rangle^L . \langle \eta \rangle \right)^{-1} \left(\langle \eta \rangle^L \triangleright v^{-1} \right) \tau\left(\langle \eta \rangle^L \triangleleft v^{-1} . \left(\langle \eta \rangle^L \triangleleft v^{-1} \right)^R \right) \right) \quad ...(2)$$

The coevaluation map, which is a morphism in C is defined by:

$$\mathrm{coev}(1) = \sum_{\xi \in basis} \xi \bar{\triangleleft} \tau(\langle \xi \rangle^L . \langle \xi \rangle^{-1} \otimes \hat{\xi}),$$

where 'is a corresponding dual basis of $\xi \in V_s$, $\forall_s \in$ M.

The dependence on the choice of representatives was explained as follows: For a given subgroup G of a group X we can choose different sets of representatives M and <u>M</u> for the left cosets. These are related by an arbitrary function γ : G\X→ G, so that if s ∈ M then γ ([s])s ∈ M. We now have two sets of binary operations "cocycles" and left "actions". We write these as · and · : G\X × G\X →G\X, τ and τ : G\X × G\X → G and and : G\X →G for M and <u>M</u>, respectively. The right action in both cases is the canonical right action by G on G\X. The following equations connect the different operation:

$$t \triangleright u = \gamma(t)(t \triangleright u)\gamma(t \triangleleft u)^{-1}, \quad s \underline{\cdot} t = (s \triangleleft \gamma(t)).t,$$

$$\underline{\tau}(s,t) = \gamma(s)(s \triangleright \gamma(t))\, \tau(s \triangleleft \gamma(t), t)\, \gamma((s \triangleleft \gamma(t)).t)^{-1}$$

...(3)

$$\xi(p.u) = (p.s,t)t = (p \triangleright \tau(s,t))\tau(p \triangleleft \tau(s,t), s.t).$$

Instead of representatives, we should have phrased the last equations in terms of cosets [s] = Gs.

The tensor structure given by M,in the category $\mathcal{C}$, shall be called ⊗ as usual and the tensor structure given by <u>M</u> shall be called ⊗. The map F_{VW} : V⊗W → V⊗W is defined by F(ξ⊗η) = ξ r(⟨η⟩)⊗ η and this gives an equivalence of the two tensor product structures. To check that F is a morphism in $\mathcal{C}$ we note that F preserve the G\X grade from the equation for the binary operation given in (3). For the G-action we use

$$F(\xi \otimes \eta) \bar{\triangleleft} u = \xi \bar{\triangleleft} \gamma\left(\langle \eta \rangle \right) \left(\langle \eta \rangle \triangleright u \right) \otimes \eta \bar{\triangleleft} u =$$

$$\xi \bar{\triangleleft} \left(\langle \eta \rangle \underline{\triangleright} u \right) \gamma\left(\langle \eta \rangle \triangleleft u \right) \otimes \eta \bar{\triangleleft} u = F\left(\left(\xi \underline{\otimes} \eta \right) \bar{\triangleleft} u \right)$$

Now, we consider morphisms θ : V→V and ϕ : W→ in $\mathcal{C}$. It was shown that the following diagram commutes:

$$\begin{tikzcd} \underline{V}\otimes W \arrow[r, "\theta\otimes\phi"] \arrow[d, "F_{VW}"] & \tilde{\underline{V}}\otimes\tilde{W} \arrow[d, "F_{\tilde{V}\tilde{W}}"] \\ V\otimes W \arrow[r, "\theta\otimes\phi"] & \tilde{V}\otimes\tilde{W} \end{tikzcd}$$

To check that F preserve the associators, it is sufficient to show that the following diagram commutes:

$$\begin{tikzcd} (\underline{U\otimes V})\underline{\otimes} W \arrow[r, "\underline{\Phi}"] \arrow[d, "(F_{UV}\otimes id)\circ F_{U\otimes V,W}"] & (U\underline{\otimes} V\underline{\otimes} W) \arrow[d, "(id\otimes F_{VW})\circ F_{U,V\otimes W}"] \\ (U\otimes V)\otimes W \arrow[r, "\Phi"] & U\otimes(V\otimes W) \end{tikzcd}$$

which can be obtained directly from the relation between τ and τ in (3). The unit objects are the same for both tensor structures. It was also shown that the following diagrams

$$\begin{tikzcd} V \arrow[r, "\underline{l}"] \arrow[d, "id"] & V\underline{\otimes} k \arrow[d, "F_{Vk}"] \\ V \arrow[r, "\underline{l}"] & V\otimes k \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\underline{r}"] \arrow[d, "id"] & k\underline{\otimes} V \arrow[d, "F_{kV}"] \\ V \arrow[r, "r"] & k\otimes V \end{tikzcd}$$

For the dual of on object V in $\mathcal{C}$, we use the assumption on the rigid tensor category structure, which are that the coset representatives include e and have right inverse. As X is a finite group, this implies that the left inverse map is 1-1, i.e. that if $s^L = t^L$ then $s = t$. The dual V' is the dual vector space. but the G\X grading and G action depend on M. Remember that if $\alpha\in V'$ annihilates all V_p for $p \neq q \in$ G\X, then it has grade q^L, but the left inverse depends on the choice of M. We call $\underline{V'}$ the dual for the coset representatives $\underline{M}$, and use $\langle\ \rangle$ and for the grading and the action on $\underline{V'}$.

It was shown, in[3], that the map H_V : $\underline{V'} \to V'$ defined by $H_V(\alpha) = \alpha \quad \gamma(q)^{-1}$ where $\alpha\in V'$ annihilates all V_p for $p \neq q \in$ G\X, is a morphism in C. Moreover, it was proved that the following diagrams commute:

$$\begin{tikzcd} \underline{V'\otimes} V \arrow[r, "\underline{eval}"] \arrow[d, "F_{V'V}\circ(H_V\otimes id)"] & k \arrow[d, "id"] \\ V'\otimes V \arrow[r, "eval"] & k \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, "\underline{coev}"] \arrow[d, "id"] & V\underline{\otimes V'} \arrow[d, "F_{V'V}\circ(id\otimes H_V)"] \\ k \arrow[r, "coev"] & V\otimes V' \end{tikzcd}$$

The algebra A in the tensor category $\mathcal{C}$ is constructed so that the group action and the grading in the definition of $\mathcal{C}$ can be combined. Consider a single object A, a vector space spanned by a basis $\delta_s\otimes u$ for $s \in M$ and $u \in G$. For any object V in $\mathcal{C}$ define a map : $V\otimes A \to V$ by $\xi \quad (\delta_s\otimes u) = \delta_{s\langle\xi\rangle}\xi \quad u$. This map is a morphism in $\mathcal{C}$ only if $\langle\xi\rangle . \langle\delta_s\otimes u\rangle = \langle\xi \quad u\rangle$ i.e.s. $\langle\delta_s\otimes u\rangle = s \quad u$ if $\langle\xi\rangle = s$. If we put $a = \langle\delta_s\otimes u\rangle$ the action of $v \in G$ is given by $(\delta_s\otimes u) \quad v = \delta_{s \quad (\alpha \quad v)} \otimes (a \quad v)^{-1} u v$.

Different choices of coset representatives

In [3] it was stated that for a given subgroup G of a group X, different sets of representatives *M* and *M* for the left cosets can be chosen and these are related by an arbitrary function :γ G \ X → G, so that if s ∈ *M* then γ ([s])s ∈ *M*. Also the binary operations $\underline{.}$: G \ X × G \ X → G \ X, τ: G \ X × G \ X → G and : G \ X × G → G for *M* were shown to be the following:

$$s.t = (s \triangleleft \gamma(t).t), \quad \underline{\tau}(s,t) = \gamma(s)(s \triangleright \gamma(t))$$

$$\tau(s \triangleleft \gamma(t).t)\gamma((s \triangleleft \gamma(t)).t)^{-1}$$

$$t \underline{\triangleright} u = \gamma(t)(t \triangleright u)\gamma(t \triangleleft u)^{-1}$$

Here we prove that for the algebras *A* and *A*, the map θ: *A* → *A* defined by $\theta_{\underline{A}}(\delta_s \otimes u) = \delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1} u$, is a morphism in the category $\mathcal{C}$. In additin, we prove that the morphism θ: *A* → *A* is an algebra map.

Proposition

For the algebras $\underline{A}$ and A, the map : $\underline{A}$ → A defined by

$$\theta_{\underline{A}}(\delta_s \otimes u) = \delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1} u,$$

where $(\delta_s \otimes u) \in \underline{A}$ and $\underline{a} = \langle \delta_s \otimes u \rangle$ is a morphism in the category $\mathcal{C}$.

Proof

We should first show that $\theta_{\underline{A}}$ preserves the grades. Let $\underline{a} = \langle \delta_s \otimes u \rangle$ then $\underline{a}$ is defined by

$$S \,\underline{.}\, \underline{\alpha} = s \underline{\triangleleft} u = s \triangleleft u,$$

but also we have

$$\left(s \triangleleft \gamma(\underline{a})\right) . \left\langle \theta_{\underline{A}}(\delta_s \otimes u) \right\rangle = \left(s \triangleleft \gamma(\underline{a})\right) \triangleleft \gamma(\underline{a})^{-1}$$

$$u = s \triangleleft \gamma(\underline{a}) \gamma(\underline{a})^{-1} u = s \triangleleft u,$$

so it preserves the grades. Now we need to check that it also preserves the actions, i.e.

$$\theta_{\underline{A}}\left((\delta_s \otimes u) \bar{\underline{\triangleleft}} v\right) = \theta_{\underline{A}}(\delta_s \otimes u) \bar{\triangleleft} v$$

To calculate the left hand side we need to calculate the following:

$$(\delta_s \otimes u) \bar{\underline{\triangleleft}} v = \delta_{s \underline{\triangleleft} (\underline{a} \underline{\triangleright} v)} \otimes (\underline{a} \underline{\triangleright} v)^{-1} uv =$$

$$\delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright u)\gamma(\underline{a} \triangleleft v)^{-1}} \otimes \gamma(\underline{a} \triangleleft v)(\underline{a} \triangleright v)^{-1} \gamma(\underline{a})^{-1} uv$$

so

L.H.S.

$$= \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright u)\gamma(\underline{a} \triangleleft u)^{-1}\gamma(\underline{a} \triangleleft u)} \otimes \gamma(\underline{a} \triangleleft v)^{-1} \gamma(\underline{a} \triangleleft v)(\underline{a} \triangleright v)^{-1} \gamma(\underline{a})^{-1} uv$$

$$= \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright v)} \otimes (\underline{a} \triangleright v)^{-1} \gamma(\underline{a})^{-1} uv,$$

Now we calculate the right hand side

R.H.S. =

$$\theta_{\underline{A}}(\delta_s \otimes u) \bar{\triangleleft} v = \left(\delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1} u\right) \bar{\triangleleft} u$$

$$= \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright u)} \otimes (\underline{a} \triangleright v)^{-1} \gamma(\underline{a})^{-1} uv,$$

which shows that θ preserves the actions.

In [3], the morphism $F_{V,W}$: V⊗W → V⊗W for V,W ∈ $\mathcal{C}$ and ⊗ is the tensor structure given by $\underline{M}$, was defined by F(ξ⊗η) =ξ γ(⟨⊗⟩)⊗η. This morphism will be used in the next proposition.

Proposition

For the algebras $\underline{A}$ and A, the morphism θ: $\underline{A}$ → A is an algebra map. Note that we have to be

$$\begin{tikzcd}
\underline{A}\otimes\underline{A} \arrow[r, "\mu"] \arrow[d, "\theta\otimes\theta"'] & \underline{A} \arrow[dd, "\theta"] \\
A\otimes A \arrow[d, "F_{AA}"'] & \\
A\otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

careful about just what this means, as there are two different tensor products. We mean that the following diagram commutes:

Proof

For the elements $(\delta_s \otimes u)$ and $(\delta_t \otimes v)$ in the algebra $\underline{A}$ we have

$$\underline{\mu}((\delta_s \otimes u)\otimes \delta_t \otimes v)) = \delta_{t,s \triangleleft \underline{u}} \delta_{s \triangleleft \tau(\underline{a},\underline{b})} \otimes \underline{\tau}(\underline{a},\underline{b})^{-1} uv,$$

where

$$\underline{a} = \langle \delta_s \otimes u \rangle \text{and} \underline{b} = \langle \delta_t \otimes v \rangle$$

so

$$\theta\left(\underline{\mu}((\delta_s \otimes u)\otimes \delta_t \otimes v))\right) = \delta_{t,s \triangleleft \underline{u}} \delta_{s \triangleleft \tau(\underline{a},\underline{b})\gamma(\underline{a}.\underline{b})}$$

$$\otimes \gamma(\underline{a}.\underline{b})^{-1} \underline{\tau}(\underline{a},\underline{b})^{-1} uv,$$

But we know that

$$\tau(\underline{a},\underline{b}) = \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b}))\, \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b}) \gamma((\underline{a} \triangleleft \gamma(\underline{b})).\underline{b})^{-1}$$

and $\underline{a}.\underline{b} = (\underline{a} \triangleleft \gamma(\underline{b})).\underline{b},$

So

$$\underline{\tau}(\underline{a},\underline{b})\gamma(\underline{a}.\underline{b}) = \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b}))\, \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b})$$

Thus, one direction of the diagram is given by the following equation:

$$\theta\left(\underline{\mu}((\delta_s \otimes u)\otimes \delta_t \otimes u))\right) = \delta_{t,s \triangleleft \underline{u}} \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b}))}$$

$$_{\tau(\underline{a} \triangleleft \gamma(\underline{b}),\underline{b})} \otimes \tau(\underline{a} \triangleleft \gamma(\underline{b}),\underline{b})^{-1}(\underline{a} \triangleright \gamma(\underline{b}))^{-1}\gamma(\underline{a})^{-1}uv,$$

Now to calculate the other direction of the diagram we do the following calculation:

$$\theta(\delta_s \otimes \underline{u}) \otimes \theta(\delta_t \otimes v) = (\delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1}\underline{u}) \underline{\otimes} (\delta_{t \triangleleft \gamma(\underline{b})} \otimes \gamma(\underline{b})^{-1} v)$$

Applying the map F_{AA} to the above equation gives

$$F_{AA}\left(\theta(\delta_s \otimes u) \underline{\otimes}\, \theta(\delta_t \otimes v)\right) = \left(\delta_{s \triangleleft \gamma(\underline{a})} \otimes \gamma(\underline{a})^{-1} u\right.$$

$$\left.\bar{\triangleleft}\, \gamma(\underline{b})\right) \otimes \left(\delta_{t \triangleleft \gamma(\underline{b})} \otimes \gamma(\underline{b})^{-1} v\right)$$

$$= \left(\delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b}))} \otimes (\underline{a} \triangleright \gamma(\underline{b})\right)^{-1} \gamma(\underline{a})^{-1} u \gamma(\underline{b}))$$

$$\otimes \left(\delta_{t \triangleleft \gamma(\underline{b})} \otimes \gamma(\underline{b})^{-1} v\right)$$

Therefore, the other direction of the diagram is given by the following equation:

$$\mu\left(F_{AA}\left(\theta(\delta_s \otimes u) \underline{\otimes}\, \theta(\delta_t \otimes v)\right)\right) =$$

$$\delta_{t \triangleleft \gamma(\underline{b}), s \triangleleft u \gamma(\underline{b})} \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b})) \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b})}$$

$$\otimes \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b})^{-1} (\underline{a} \triangleright \gamma(\underline{b}))^{-1} \gamma(\underline{a})^{-1} u \gamma(\underline{b}) \gamma(\underline{b})^{-1} v$$

$$= \delta_{t,s \triangleleft u} \delta_{s \triangleleft \gamma(\underline{a})(\underline{a} \triangleright \gamma(\underline{b})) \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b})}$$

$$\otimes \tau(\underline{a} \triangleleft \gamma(\underline{b}), \underline{b})^{-1} (\underline{a} \triangleright \gamma(\underline{b}))^{-1} \gamma(\underline{a})^{-1} uv,$$

which is the same as the first direction and this completes the proof.

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